10 Instrumental Variables

10.1 Derivation of Equation 10.4

$$\hat{f}(x) = h^{-1} \int k\left(\frac{x-z}{h}\right) d\hat{F}(z) = h^{-1} \int k\left(\frac{x-z}{h}\right) \frac{d\hat{F}(z)}{dz} dz.$$

Following Li and Racine (2007, page 679), we use the Dirac delta function, $\delta(x)$, where $\delta(x) = 0$ for $x \neq 0$, $\delta(0) = \infty$ and $\int \delta(x)dx = 1$ to differentiate the ECDF. Note that $d1\{x \geq 0\}/dx = \delta(x)$. With this notation we have

$$\begin{aligned}
\hat{f}(x) =& h^{-1} \int k\left(\frac{x-z}{h}\right) \frac{d\hat{F}(z)}{dz} dz \\
=& h^{-1} \int k\left(\frac{x-z}{h}\right) \left(n^{-1} \sum_{i=1}^{n} d1\{(z-x_i) \geq 0\}/dz\right) dz \\
=& h^{-1} \int k\left(\frac{x-z}{h}\right) \left(n^{-1} \sum_{i=1}^{n} \delta(z-x_i)\right) dz \\
=& (nh)^{-1} \sum_{i=1}^{n} \int k\left(\frac{x-z}{h}\right) \delta(z-x_i) dz.
\end{aligned}$$

Using the change of variables $\eta = (x-z)/h$ we have $hd\eta = dz$ and $z = x - h\eta$ we now have

$$\begin{aligned}
\hat{f}(x) =& (nh)^{-1} \sum_{i=1}^{n} \int k(\eta)\, \delta(x - x_i - h\eta) dz \\
=& (nh)^{-1} \sum_{i=1}^{n} \int k(\eta)\, \delta((x-x_i)/h - \eta) h dz \\
=& (nh)^{-1} \sum_{i=1}^{n} \int k(\eta)\, \delta((x-x_i)/h - \eta) d\eta.
\end{aligned}$$

From Li and Racine (2007, Exercise A.11), it follows that for any measurable function $w(\cdot)$, $\int \delta(\eta - a)w(\eta)d\eta = w(a)$. Thus, we have

$$\begin{aligned}\hat{f}(x) =&(nh)^{-1}\sum_{i=1}^{n}\int k\left(\eta\right)\delta((x-x_i)/h-\eta)d\eta \\ =&(nh)^{-1}\sum_{i=1}^{n}\int k\left(\frac{x_i-x}{h}\right),\end{aligned}$$

given that the Dirac delta function and the kernel are symmetric.